ON DUALITY THEOREMS FOR ROBUST OPTIMIZATION PROBLEMS

Gue Myung Lee* and Moon Hee Kim**

Abstract. A robust optimization problem, which has a maximum function of continuously differentiable functions as its objective function, continuously differentiable functions as its constraint functions and a geometric constraint, is considered. We prove a necessary optimality theorem and a sufficient optimality theorem for the robust optimization problem. We formulate a Wolfe type dual problem for the robust optimization problem, which has a differentiable Lagrangean function, and establish the weak duality theorem and the strong duality theorem which hold between the robust optimization problem and its Wolfe type dual problem. Moreover, saddle point theorems for the robust optimization problem are given under convexity assumptions.

1. Introduction

Consider the standard nonlinear programming problem with inequality constraints and a geometric constraint.

$$\text{(P)} \quad \inf_{x \in \mathbb{R}^n} \{ f(x) : g_i(x) \leq 0, \ i = 1, \cdots, m, \ x \in C \},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions, and C is a closed convex subset of $\mathbb{R}^n$. The problem in the face of data uncertainty in the objective function and the constraints can be captured by the following nonlinear programming problem:

$$\text{(UP)} \quad \inf_{x \in \mathbb{R}^n} \{ f(x, u) : g_i(x, v_i) \leq 0, \ i = 1, \cdots, m, \ x \in C \},$$

where u, v_i are uncertain parameters and $u \in U$, $v_i \in V_i, i = 1, \cdots, m$ for some convex compact sets $U \subset \mathbb{R}^p, V_i \subset \mathbb{R}^q, i = 1, \cdots, m$, respectively and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \cdots, m$ are continuously differentiable. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty ([1] - [4], [6]), associates with the uncertain program (UP) its robust counterpart [5],

$$(\text{RP}) \quad \inf_{x \in \mathbb{R}^n} \{\max_{u \in U} f(x, u) : g_i(x, v_i) \leq 0, \ \forall v_i \in V_i, \ \ i = 1, \cdots, m, \ x \in C\},$$

where the uncertain objective function and constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets U, V_i, $i = 1, \cdots, m$. Recently, Jeyakumar, Li and Lee [7] established necessary optimality theorems and robust duality theorems for a generalized convex programming problem in the face of data uncertainty. Kuroiwa and Lee [9] extended the necessary optimality theorem to a multiobjective robust optimization problem. Futhermore, Kim [8] extended the robust duality theorems to a multiobjective robust optimization problem.

In this paper, we consider a robust optimization problem, which has a maximum function of continuously differentiable functions as its objective function, continuously differentiable functions as its constraint functions and a geometric constraint. We prove necessary and sufficient optimality theorems for the robust optimization problem. We formulate a Wolfe type dual problem for the robust optimization problem, which has a differentiable Lagrangean function, and establish the weak duality theorem and the strong duality theorem which hold between the robust optimization problem and its Wolfe type dual problem. Moreover, saddle point theorems for the robust optimization problem are given under convexity assumptions.

2. Robust optimality theorems

In this section, we provide necessary, and sufficient optimality conditions for the uncertain optimization problem (UP) by using its robust counterpart (RP). To begin with, we recall that the robust feasible set F is defined by

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \ \forall v_i \in V_i, \ i = 1, \cdots, m, \ x \in C\}.$$

We say that x^* is a robust solution of (UP) if x^* is a solution of (RP), that is, $x^* \in F$ and $\max_{u \in U} f(x, u) \geq \max_{u \in U} f(x^*, u)$ for any $x \in F$. We denote $\nabla_1 g$ the derivative of g with respect to the first variable.

DEFINITION 2.1. Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$. Let $N_C(x) = \{v \in \mathbb{R}^n \mid v^T(y - x) \leq 0 \text{ for all } y \in C\}$. Then $N_C(x)$ is called the normal cone to C at x.

LEMMA 2.2. [12] *Let Θ be a nonempty, compact topological space and let $h : \mathbb{R}^n \times \Theta \to \mathbb{R}$ be such that $h(\cdot, \theta)$ is differentiable for every $\theta \in \Theta$ and $\nabla_1 h(x, \theta)$ is continuous on $\mathbb{R}^n \times \Theta$. Let $\phi(x) = \sup_{\theta \in \Theta} h(x, \theta)$. Define $\bar{\Theta}$ to be $\bar{\Theta}(x) := \arg\max_{\theta \in \Theta} h(x, \theta)$. Then the function $\phi(x)$ is locally Lipschitz continuous, directionally differentiable and*

$$\phi'(x, d) = \sup_{\theta \in \bar{\Theta}(x)} \nabla_1 h(x, \theta)^T d,$$

where $\phi'(x, d) = \lim_{t \to 0+} \frac{\phi(x + td) - \phi(x)}{t}$.

Let $\bar{x} \in F$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in V_j \text{ s.t. } g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Since $\bar{x} \in F$, $J_1(\bar{x}) = \{j \in J \mid \max_{v_j \in V_j} g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = \{j \in J \mid \max_{v_j \in V_j} g_j(\bar{x}, v_j) < 0\}$.

Now we say that an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$ for (RP) if there exists $\hat{x} \in C$ such that for any $j \in J_1(\bar{x})$ and any $v_j \in V_j$,

$$\nabla_1 g_j(\bar{x}, v_j)^T(\hat{x} - \bar{x}) < 0.$$

Now we present a necessary optimality theorem for a robust solution of (UP). Following the approaches of the proofs for Theorem 3.1 in [7] and Theorem 3.7 in [9], we can prove the following theorem. For the completeness, we give the proof for the following theorem.

THEOREM 2.3. *Let $\bar{x} \in F$ be a robust solution of (UP). Suppose that $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$ are concave on V_j, $j = 1, \cdots, m$. Then there exist $\lambda \geq 0$, $\mu_j \geq 0$, $j = 1, \cdots, m$, not all zero, $\bar{u} \in U$ and $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ such that*

$$0 \in \lambda \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + N_C(\bar{x}),$$

$$f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),$$

$$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.$$

Moreover, if we assume that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$, then

$$(2.1) \qquad 0 \in \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + N_C(\bar{x}),$$

$$(2.2) \qquad f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),$$

$$(2.3) \qquad \mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.$$

Proof. Assume that $\max_{v_j \in V_j} g_j(\bar{x}, v_j) < 0$, $j = 1, \cdots, m$, and $J_1(\bar{x}) = \emptyset$. Then $\bar{x} \in \text{int} F$, where $\text{int} F$ is the interior of F. Let $\psi(x) = \max_{u \in U} f(x, u)$. Then $U^0 = \{u \in U \mid f(\bar{x}, u) = \psi(\bar{x})\}$. Then U^0 is convex and compact. By Lemma 2.2, for any $d \in \mathbb{R}^n$,

$$\psi'(\bar{x}, d) = \max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T d.$$

Suppose to the contrary that there exists $x^* \in C$ such that

$$\max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x^* - \bar{x}) < 0.$$

Then there exists $\delta > 0$ such that for any $t \in (0, \delta)$,

$$\psi(\bar{x} + t(x^* - \bar{x})) < \psi(\bar{x}).$$

Since $\bar{x} \in \text{int} F$, this contradicts the optimality at $\bar{x}$. Thus there does not exist $x \in C$ such that

$$\max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x - \bar{x}) < 0.$$

We assume that $J_1(\bar{x}) \neq \emptyset$. Let $\varphi_j(x) = \max_{v_j \in V_j} g_j(x, v_j)$, $j = 1, \cdots, m$ and $V_j^0 = \{v_j \in V_j \mid g_j(\bar{x}, v_j) = \varphi_j(\bar{x})\}$, $j = 1, \cdots, m$. Then V_j^0 is convex and compact. By Lemma 2.2, for any $d \in \mathbb{R}^n$,

$$\varphi_j'(\bar{x}, d) = \max_{v_j \in V_j^0} \nabla_1 g_j(\bar{x}, v_j)^T d, \ j = 1, \cdots, m.$$

Assume to the contrary that the following system has a solution $x^* \in C$;

$$\max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x^* - \bar{x}) < 0,$$
$$\max_{v_j \in V_j^0} \nabla_1 g_j(\bar{x}, v_j)^T (x^* - \bar{x}) < 0, \ j \in J_1(\bar{x}).$$

Thus the following system has a solution $x^* \in C$;

$$\psi'(\bar{x}; x^* - \bar{x}) < 0,$$
$$\varphi_j'(\bar{x}; x^* - \bar{x}) < 0, \ j \in J_1(\bar{x}).$$

Since $\varphi_i(x)$, $i \in J_2(\bar{x})$ is continuously differentiable, there exists $\delta > 0$ such that for any $t \in (0, \delta)$,

$$\bar{x} + t(x^* - \bar{x}) \in C,$$
$$\psi(\bar{x} + t(x^* - \bar{x})) < \psi(\bar{x}),$$
$$\varphi_j(\bar{x} + t(x^* - \bar{x})) < \varphi_j(\bar{x}) = 0, \ j \in J_1(\bar{x}),$$
$$\varphi_i(\bar{x} + t(x^* - \bar{x})) < 0, \ i \in J_2(\bar{x}).$$

This contradicts the optimality of $\bar{x}$. Thus the following system has no solution $x \in C$:

$$\max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x - \bar{x}) < 0,$$
$$\max_{v_j \in V_j^0} \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x}) < 0, \ j \in J_1(\bar{x}).$$

Let $h_0(x) = \max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x - \bar{x})$ and $h_j(x) = \max_{v_j \in V_j^0} \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x})$, $j \in J_1(\bar{x})$. Then h_1 and h_j, $j \in J_1(\bar{x})$ are convex. Thus, by Generalized Gordan alternative theorem in [10, p. 65], there exist $\lambda \geq 0$, $\mu_j \geq 0$, $j \in J_1(\bar{x})$, not all zero, such that for all $x \in C$,

$$\lambda \max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T (x - \bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \max_{v_j \in V_j^0} \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x}) \geq 0,$$

that is,

$$\inf_{x \in C} \max_{u \in U^0, v_j \in V_j^0} \left[\lambda \nabla_1 f(\bar{x}, u)^T (x - \bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x}) \right] \geq 0.$$

We can check that $u \mapsto \nabla_1 f(\bar{x}, u)^T (x - \bar{x})$ and $v_j \mapsto \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x})$, $j \in J_1(\bar{x})$ are concave on U^0 and V_j^0, respectively (See the proof of Theorem 3.1 in [7]). So, by min-max theorem [11, Corollary 37.3.2],

$$\max_{u \in U^0, v_j \in V_j^0} \inf_{x \in C} \left[\lambda \nabla_1 f(\bar{x}, u)^T (x - \bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, v_j)^T (x - \bar{x}) \right] \geq 0.$$

Thus there exists $\bar{u}_i \in U$ and $\bar{v}_j \in V_j$, $j \in J_1(\bar{x})$ such that for any $x \in C$,

$$\lambda \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}) \geq 0.$$

From (EMFCQ), λ can not be 0. Thus we may assume that $\lambda = 1$. Hence for any $x \in C$,

$$\nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}) \geq 0,$$

that is,

$$-\nabla_1 f(\bar{x}, \bar{u}) - \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \in N_C(\bar{x}).$$

Therefore

$$0 \in \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + N_C(\bar{x}).$$

$\square$

Now we give a sufficient optimality theorem for (UP).

THEOREM 2.4. *Let* $\bar{x} \in F$. *Suppose that there exist* $\mu_j \geq 0$, $j = 1, \cdots, m$, $\bar{u} \in U$ *and* $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ *such that*

$$0 \in \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + N_C(\bar{x}),$$

$$f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),$$

$$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.$$

If $f(\cdot, \bar{u})$ *and* $g_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$, *are convex on* $\mathbb{R}^n$, *then* $\bar{x} \in F$ *is a robust solution of* (UP).

Proof. For any $x \in F$,

$$\max_{u \in U} f(x, u) - f(\bar{x}, \bar{u})$$
$$\geq f(x, \bar{u}) - f(\bar{x}, \bar{u})$$
$$\geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x})$$
$$\geq - \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \Big]^T (x - \bar{x})$$
$$\geq - \sum_{j=1}^{m} \bar{\mu}_j \Big[g_j(x, \bar{v}_j) - g_j(\bar{x}, \bar{v}_j) \Big]$$
$$= - \sum_{j=1}^{m} \bar{\mu}_j g_j(x, \bar{v}_j)$$
$$\geq 0.$$

Hence $\max_{u \in U} f(x, u) \geq \max_{u \in U} f(\bar{x}, u)$ for any $x \in F$, and so $\bar{x} \in F$ is a robust solution of (UP). $\square$

3. Robust duality theorems

Now we formulate a Wolfe type robust dual problem (WD) for (RP).

(WD)   maximize $f(x, u) + \sum_{j=1}^{m} \mu_j g_j(x, v_j)$

subject to $0 \in \nabla_1 f(x, u) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(x, v_j) + N_C(x),$

$\mu_j \geq 0, \quad u \in U, \ v_j \in V_j, \ j = 1, \cdots, m.$

Let $V = V_1 \times \cdots \times V_m$.

Now we establish duality theorems (weak duality theorem, strong duality theorem) which hold between (RP) and (WD).

THEOREM 3.1. *(Weak Duality) Let $x \in \mathbb{R}^n$ be feasible for (RP) and $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}) \in \mathbb{R}^n \times U \times V \times \mathbb{R}^m$ be feasible for (WD). Suppose that $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j), \ j = 1, \cdots, m$ are convex, then*

$$\max_{u \in U} f(x, u) \geq f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j).$$

Proof. Let x be feasible for (RP) and $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ be feasible for (WD). Then there exists $\bar{\xi} \in N_C(\bar{x})$ such that

$$\nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + \bar{\xi} = 0.$$

Then we have,

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) - \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$$

$$\geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}) - \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$$

$$\geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \Big[g_j(x, \bar{v}_j) - g_j(\bar{x}, \bar{v}_j) \Big]$$

$$\geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x})$$

$$= \left[\nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \right]^T (x - \bar{x})$$

$$= (-\bar{\xi})^T (x - \bar{x}).$$

Since $\bar{\xi} \in N_C(\bar{x})$ and $x \in C$, $(-\bar{\xi})^T(x - \bar{x}) \geq 0$. Hence $f(x, \bar{u}) \geq f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$. Therefore, $\max_{u \in U} f(x, u) \geq f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$.
$\square$

THEOREM 3.2. (Strong Duality) Let $\bar{x}$ be a robust solution of (UP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$. Then, there exist $(\bar{u}, \bar{v}, \bar{\mu})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is feasible for (WD) and the objective values of (RP) and (WD) are equal. If $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are convex, $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$, $j = 1, \cdots, m$ are concave on V_j, then $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a solution of (WD).

Proof. Since $\bar{x}$ is a robust solution of (UP) at which the Extended Mangasarian-Fromovitz constraint qualification holds, then by Theorem 2.3, there exist $\bar{\mu}_j \geq 0$, $j = 1, \cdots, m$, $\bar{u} \in U$ and $\bar{v}_j \in V_j$, $j = 1, \cdots, m$, such that

$$0 \in \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + N_C(\bar{x}),$$

$$f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),$$

$$\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.$$

Thus $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is feasible for (WD) and the objective values of (RP) and (WD) are equal. Moreover, $\max_{u \in U} f(\bar{x}, u) = f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$. It follows from weak duality (Theorem 3.1) that for any feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu})$ for (WD),

$$\begin{aligned}
\max_{u \in U} f(\bar{x}, u) &= f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \\
&\geq f(\widetilde{x}, \widetilde{u}) + \sum_{j=1}^{m} \widetilde{\mu}_j g_j(\widetilde{x}, \widetilde{v}_j).
\end{aligned}$$

Hence $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a solution of (WD). $\square$

4. Saddle-point theorems

In this section, we prove saddle-point theorems for the robust optimization problem (RP). Let

$$L(x, v, \mu) = \max_{u \in U} f(x, u) + \sum_{j=1}^{m} \mu_j g_j(x, v_j),$$

where $x \in \mathbb{R}^n$, $u \in U$, $v_j \in V_j$, and $\mu \in \mathbb{R}_+^m$. Then, a point $(\bar{x}, \bar{v}, \bar{\mu}) \in \mathbb{R}^n \times V \times \mathbb{R}_+^m$ is said to be a saddle-point for (RP) if

$$L(x, \bar{v}, \bar{\mu}) \geq L(\bar{x}, \bar{v}, \bar{\mu}) \geq L(\bar{x}, v, \mu),$$

for all $x \in C$, $v \in V$, $\mu \in \mathbb{R}_+^m$, where $V = V_1 \times \cdots \times V_m$.

THEOREM 4.1. *Let $\bar{x}$ be a feasible solution of (RP) and let $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ satisfy (2.1), (2.2) and (2.3). Suppose that $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are convex. Then $(\bar{x}, \bar{v}, \bar{\mu})$ is a saddle-point for (RP).*

Proof. Assume that (2.1), (2.2) and (2.3) holds. From (2.1), there exists $\bar{\xi} \in N_C(\bar{x})$ such that

$$\nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) + \bar{\xi} = 0.$$

Let $x \in \mathbb{R}^n$ be any fixed. Since $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are convex,

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}),$$
$$g_j(x, \bar{v}_j) - g_j(\bar{x}, \bar{v}_j) \geq \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}).$$

Since $\bar{\mu}_j \geq 0$, $j = 1, \cdots, m$,

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq \nabla_1 f(\bar{x}, \bar{u})^T (x - \bar{x}),$$
$$\bar{\mu}_j g_j(x, \bar{v}_j) - \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \geq \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}), \ j = 1, \cdots, m.$$

Summing up all these inequalities, for any $x \in C$

$$\left\{ f(x, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x, \bar{v}_j) \right\} - \left\{ f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \right\}$$

$$\geq \left\{ \nabla_1 f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \right\}^T (x - \bar{x})$$

$$= -\bar{\xi}^T (x - \bar{x})$$

$$\geq 0.$$

From (2.2), for any $x \in C$,

$$
\begin{aligned}
\max_{u \in U} f(x, u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x, \bar{v}_j) \quad &\geq \quad f(x, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(x, \bar{v}_j) \\
&\geq \quad f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \\
&= \quad \max_{u \in U} f(\bar{x}, u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j),
\end{aligned}
$$

that is, $L(x, \bar{v}, \bar{\mu}) \geq L(\bar{x}, \bar{v}, \bar{\mu})$ for any $x \in C$. Now, since $\bar{x}$ is feasible for (RP), it follows from (2.3) that

$$
\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) - \sum_{j=1}^{m} \mu_j g_j(\bar{x}, v_j) \geq 0
$$

for any $\mu \in \mathbb{R}^m_+$ and $v_j \in V_j$. Thus

$$
f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) - \left\{ f(\bar{x}, \bar{u}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, v_j) \right\} \geq 0,
$$

and hence for any $\mu \in \mathbb{R}^m_+$ and $v_j \in V_j$,

$$
\max_{u \in U} f(\bar{x}, u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) - \left\{ \max_{u \in U} f(\bar{x}, u) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, v_j) \right\} \geq 0,
$$

that is, $L(\bar{x}, \bar{v}, \bar{\mu}) \geq L(\bar{x}, v, \mu)$. Therefore, $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu})$ is a saddle-point of (RP). $\qquad \square$

THEOREM 4.2. *If there exists $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{v}, \bar{\mu})$ is a saddle-point for (RP), then $\bar{x}$ is a robust solution for (UP).*

Proof. Let $(\bar{x}, \bar{v}, \bar{\mu})$ be a saddle-point for (RP). From the right inequality of saddle-point conditions,

$$
\max_{u \in U} f(\bar{x}, u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \geq \max_{u \in U} f(\bar{x}, u) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, v_j)
$$

for any $\mu \in \mathbb{R}^m_+$ and $v_j \in V_j$. Thus

$$
\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \geq \sum_{j=1}^{m} \mu_j g_j(\bar{x}, v_j)
$$

for any $\mu \in \mathbb{R}^m_+$ and $v_j \in V_j$. Letting $\mu = 0$ in the last inequality, $\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \geq 0$. Letting $\mu = 2\bar{\mu}$ and $v_j = \bar{v}_j$ in the last inequality

$\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \le 0$. Therefore, $\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0$. So, from the left inequality of saddle-point conditions, we have, for any feasible solution x for (RP), $\max_{u \in U} f(x, u) \ge \max_{u \in U} f(\bar{x}, u)$. Hence $\bar{x}$ is a robust solution of (UP). $\qquad\square$

References

[1] D. Bertsimas and D. Brown, *Constructing uncertainty sets for robust linear optimization*, Oper. Res. **57** (2009), 1483-1495.

[2] A. Ben-Tal and A. Nemirovski, *Robust optimization-methodology and applications*, Math. Program. Ser. B **92** (2002), 453-480.

[3] A. Ben-Tal and A. Nemirovski, *A selected topics in robust convex optimization*, Math. Program. Ser. B **112** (2008), 125-158.

[4] D. Bertsimas, D. Pachamanova, and M. Sim, *Robust linear optimizaion under general norms*, Oper. Res. Lett. **32** (2004), 510-516.

[5] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski, *Robust optimization*, Princeton Series in Applied Mathematics, 2009.

[6] V. Jeyakumar, G. Li, and G. M. Lee, *A robust von Neumann minimax theorem for zero-sum games under bounded payoff uncertainty*, Oper. Res. Lett. **39** (2011), 109-114.

[7] V. Jeyakumar, G. Li, and G. M. Lee, *Robust duality for generalized convex programming problems under data uncertainty*, Nonlinear Analysis **75** (2012), 1362-1373.

[8] M. H. Kim, *Robust duality for generalized invex programming problems*, Commun. Korean Math. Soc. **28** (2013), 419-423.

[9] D. Kuroiwa and G. M. Lee, *On robust multiobjective optimization*, Vietnam J. Math. **40** (2012), 305-317.

[10] O. L. Mangasarian, *Nonlinear Programming*, SIAM, Philadelphia, 1994.

[11] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N. J., 1970.

[12] A. Shapiro, D. Dentcheva, and A. Ruszczynski, *Lectures on Stochastic Programming: Modeling and Theory*, SIAM, Philadelphia, 2009.

Department of Applied Mathematics
Pukyong National University
Busan 608-737, Republic of Korea
E-mail: gmlee@pknu.ac.kr

**

School of Free Major
Tongmyong University
Busan 608-711, Republic of Korea
E-mail: mooni@tu.ac.kr